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AUTHOR(S):

SUGA, Shuichi

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## Relative invariants and irreducible highest weight modules

熊本電波高専 菅 修一(SUGA Shuichi)

### 1. Introduction.

It is well known that finite dimensional irreducible  $\mathfrak{sl}(2, \mathbb{C})$ -modules are constructed in the following way. Let  $\{e, h, f\}$  be a standard basis of  $\mathfrak{sl}(2, \mathbb{C})$  so that their Lie brackets are given by

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h \quad (1.1)$$

Fix a nonnegative integer  $n$ . Let  $V = \sum_{i=0}^n \mathbb{C}x^i$  be the vector space of polynomials of  $x$  whose degrees are at most  $n$ . Then the following correspondence defines an irreducible representation of  $\mathfrak{sl}(2, \mathbb{C})$  on  $V$ :

$$\left\{ \begin{array}{ll} e & \rightarrow x^2 \frac{d}{dx} - nx \\ h & \rightarrow 2x \frac{d}{dx} - n \\ f & \rightarrow -\frac{d}{dx} \end{array} \right. \quad (1.2)$$

It is easy to see that  $n$  is highest weight of  $V$  and  $x^n$  is a highest weight vector.

If one replace  $n$  by any complex number  $\lambda$  and  $V$  by  $W = \sum_{i=0}^{\infty} \mathbb{C}x^{\lambda-i}$ , then  $W$  is still an irreducible highest weight  $\mathfrak{sl}(2, \mathbb{C})$ -module with highest weight  $\lambda$  and a highest weight vector  $x^{\lambda}$ . In this note we shall generalize the above construction of irreducible highest weight representations to Lie algebras related to certain Hermitian symmetric pairs. The generalization is done by replacing  $x^{\lambda}$  by a complex power of relative invariant polynomials of prehomogeneous vector spaces attached to Hermitian symmetric spaces. This also gives a representation theoretic interpretation of the zeros of the  $b$ -function.

## 2. Construction of irreducible highest weight modules.

Let  $(G_0, K_0)$  be one of the following pairs.

1.  $G_0 = SU(n, n), \quad K_0 = S(U(n) \times U(n))$
2.  $G_0 = Sp(2n, \mathbb{R}), \quad K_0 = U(n)$
3.  $G_0 = SO(4n)^*, \quad K_0 = U(2n)$
4.  $G_0 = SO(n, 2)_0, \quad K_0 = SO(n) \times SO(2).$

Where we regard  $K_0$  as a maximal compact subgroup of the Lie group  $G_0$ . The quotient space  $G_0/K_0$  is an Hermitian symmetric space of tube type. Let  $\mathfrak{g}_0$  (resp.  $\mathfrak{k}_0$ ) be the Lie algebra of  $G_0$  (resp.  $K_0$ ), and  $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$  the Cartan decomposition of  $\mathfrak{g}_0$ . By convention we delete the subscript 0 to denote complexified Lie algebras. So we have the decomposition  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$  of the complexified Lie algebra  $\mathfrak{g}$ .

The Lie algebra  $\mathfrak{k}_0$  has the one dimensional center  $Z = \mathbb{R}z$ , where the eigenvalues of  $z$  under the adjoint action on  $\mathfrak{p}$  are  $\pm i$ .

Let

$$\mathfrak{p}^+ = \{x \in \mathfrak{p} \mid [z, x] = ix\} \text{ and } \mathfrak{p}^- = \{x \in \mathfrak{p} \mid [z, x] = -ix\}.$$

Then  $\mathfrak{p}^+$  (resp.  $\mathfrak{p}^-$ ) corresponds to the holomorphic (resp. anti-holomorphic) vector fields on the Hermitian symmetric space  $G_0/K_0$ . Let  $\mathfrak{t}_0$  be a Cartan subalgebra of  $\mathfrak{k}_0$  then  $\mathfrak{t}$  is a Cartan subalgebra of  $\mathfrak{g}$ . We set  $\mathfrak{q} = \mathfrak{k} \oplus \mathfrak{p}^+$  then  $\mathfrak{q}$  is a maximal parabolic subalgebra of  $\mathfrak{g}$ . Let  $\mathfrak{b}$  be a Borel subalgebra contained in  $\mathfrak{q}$  and contains  $\mathfrak{t}$ . Let  $G$  be the connected and simply connected complex Lie group with the Lie algebra  $\mathfrak{g}$  and  $K$  (resp.  $Q$ ) the subgroup of  $G$  corresponding to the Lie subalgebra  $\mathfrak{k}$  (resp.  $\mathfrak{q}$ ). We denote the universal covering groups of  $K$  and  $Q$  by  $\tilde{K}$  and  $\tilde{Q}$  respectively. Let  $\pi: \tilde{Q} \rightarrow Q$  be the projection homomorphism. We choose an open neighborhood  $U \subset Q$  of the identity so that there exists a section  $\sigma: U \rightarrow \tilde{Q}$ .

It is known that the pair  $(K, \mathfrak{p}^-)$  is a regular irreducible prehomogeneous vector spaces via the adjoint  $K$ -action. There exists a unique (up to constant multiple)

irreducible  $K$ -relative invariant polynomial  $f$  on  $\mathfrak{p}^-$ . By definition  $f$  is an irreducible polynomial on  $\mathfrak{p}^-$  satisfying

$$f(\text{Ad}(k)x) = \chi(k)f(x) \quad (k \in K, x \in \mathfrak{p}^-) \quad (2.1)$$

for some one dimensional character  $\chi$  of  $K$ .

We fix an arbitrary 1-dimensional character  $\lambda$  of  $\tilde{K}$ , then there exists a complex number  $m$  such that  $\lambda = \mu\chi$ . (We denote the group of one dimensional characters of  $\tilde{K}$  additively.) We extend  $\lambda$  to  $\tilde{Q}$  trivially and denote it by the same letter. We also denote the differential of  $\lambda$  by the same letter and consider it as an element of  $\mathfrak{t}^*$ .

Let  $N^-$  be the subgroup of  $G$  corresponding to  $\text{ad}(\mathfrak{p}^-)$ . Then the exponential map  $\exp: \mathfrak{p}^- \rightarrow N^-$  is a diffeomorphism. We denote its inverse by  $\log: N^- \rightarrow \mathfrak{p}^-$ . We put  $O = N^-U$ ; an open subset of  $G$ .

Let  $L(\lambda) = \{h: O \rightarrow \mathbb{C} \mid h(gq) = \lambda(\sigma(q))h(g), g \in O, q \in U\}$ . By differentiating the left translation of  $G$  we get an algebra homomorphism  $\phi: U(\mathfrak{g}) \rightarrow D(O)$ . Here  $U(\mathfrak{g})$  is the universal enveloping algebra of  $\mathfrak{g}$  and  $D(O)$  is the algebra of differential operators on  $O$  with holomorphic coefficients. The homomorphism  $\phi$  defines a  $U(\mathfrak{g})$ -action on  $L(\lambda)$ .

Using the relative invariant polynomial  $f$  on  $\mathfrak{p}^-$ , we define an element  $v^\lambda$  of  $L(\lambda)$  by the following formula:

$$v^\lambda(nq) = \lambda(\sigma(q)) f^{-2\mu}(\log(n)) \quad n \in N^-, q \in U. \quad (2.2)$$

Consider the  $\mathfrak{g}$ -module  $W(\lambda) = \phi(U(\mathfrak{g})) v^\lambda$  generated by  $v^\lambda$ .

**Theorem.**  $W(\lambda)$  is an irreducible highest weight  $\mathfrak{g}$ -module (with respect to the Borel subalgebra  $\mathfrak{b}$ ) with highest weight  $\lambda$  and a highest weight vector  $v^\lambda$ .

The proof of Theorem will appear in the forthcoming paper [6].

### 3. The case $G_0 = SU(n, n)$ , $K_0 = S(U(n) \times U(n))$ .

We illustrate the representation in Section 2. We realize the group  $G_0$  as

$$SU(n, n) = \{ g \in GL_{2n}(\mathbb{C}) \mid {}^t \bar{g} H_{2n} g = H_{2n}, \det g = 1 \} \quad (3.1)$$

where  $H_{2n} = \begin{pmatrix} 1_n & 0 \\ 0 & -1_n \end{pmatrix}$ . Then the Lie algebras  $\mathfrak{g}$ ,  $\mathfrak{k}$ ,  $\mathfrak{p}^\pm$  and  $\mathfrak{q}$  defined in Section 2 can be taken as the following form:

$$\begin{aligned} \mathfrak{g} &= \mathfrak{sl}(2n, \mathbb{C}), \quad \mathfrak{k} = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \in \mathfrak{g} \right\}, \quad \mathfrak{p}^+ = \left\{ \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \in \mathfrak{g} \right\}, \\ \mathfrak{p}^- &= \left\{ \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix} \in \mathfrak{g} \right\} \text{ and } \mathfrak{q} = \left\{ \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \in \mathfrak{g} \right\}. \end{aligned} \quad (3.2)$$

Let  $\lambda$  be any complex number we regard it as a one dimensional character of  $\mathfrak{k}$  by

$$\lambda(a) = \lambda \operatorname{tr} A \text{ for } a = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \in \mathfrak{k}. \quad (3.3)$$

From the definition we can identify  $L(\lambda)$  with the space of  $\mathbb{C}$ -valued functions on  $N^-$ . Since the latter space can be identified with the space  $\mathbb{C}(\mathfrak{p}^-)$  of  $\mathbb{C}$ -valued functions on  $\mathfrak{p}^-$  via the exponential map, we shall use this space in the following.

Then the highest weight vector  $v^\lambda$  of  $W(\lambda)$  is corresponds to  $f^\lambda$ .

Let  $(e_{jk})_{j,k=1,\dots,n}$  be the  $n \times n$  matrix units and  $(x_{jk})_{j,k=1,\dots,n}$  a standard coordinate system on  $\mathfrak{p}^-$ . Then the set of matrices

$$F_{jk} = \begin{pmatrix} 0 & 0 \\ e_{jk} & 0 \end{pmatrix} \text{ (resp. } E_{jk} = \begin{pmatrix} 0 & e_{jk} \\ 0 & 0 \end{pmatrix} \text{)}, \quad j, k = 1, 2, \dots, n$$

gives a basis of  $\mathfrak{p}^-$  (resp.  $\mathfrak{p}^+$ ).

The elements of  $\mathfrak{g}$  acts on  $\mathbb{C}(\mathfrak{p}^-)$  by the following formulas:

$$\begin{cases} E_{jk} \rightarrow \sum_{l,m=1}^n x_{lj} x_{km} \frac{\partial}{\partial x_{lm}} - \lambda x_{kj} \\ a \rightarrow \sum_{l=1}^n (x_{lj} a_{lk} - d_{jl} x_{lk}) \frac{\partial}{\partial x_{jk}} - \lambda \operatorname{tr}(A) \\ F_{jk} \rightarrow - \frac{\partial}{\partial x_{jk}} \end{cases} \quad (3.4)$$

Where  $a = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$ ,  $A = (a_{jk})$  and  $D = (d_{jk})$ . This gives a generalization of (1.2).

By the Poincaré-Birkhoff-Witt theorem we have

$$\varphi(U(g))v^\lambda = \varphi(U(p^-)U(k)U(p^+))v^\lambda = \varphi(U(p^-))v^\lambda. \quad (3.5)$$

Hence in particular we conclude that  $W(\lambda)$  consists of all the differential polynomials of  $v^\lambda$ .

#### 4. Reducibilities of generalized Verma modules

In this section we discuss reducibilities of generalized Verma modules induced from the maximal parabolic subalgebra  $q$ . This gives a representation theoretic interpretation of zeros of the  $b$ -function. We retain the notations in the previous sections.

Let  $f^*$  be the relative invariant polynomial of the prehomogeneous vector space  $(K, p^+)$  and  $f^*(D_X)$  the linear differential operator with constant coefficients defined by the equation:

$$f^*(D_X)\exp\langle \xi, x \rangle = f^*(\xi)\exp\langle \xi, x \rangle \text{ for } \xi \in p^+ \text{ and } x \in p^-. \quad (4.1)$$

where  $\langle, \rangle$  is the Killing form on  $g$ . Let  $s$  be a complex parameter then there is a polynomial  $b(s)$  such that the following differential equation holds:

$$f^*(D_X)f(x)^s = b(s)f(x)^{s-1}. \quad (4.2)$$

The polynomial  $b(s)$  is called the  $b$ -function of the relative invariant  $f$ .

Let  $\lambda$  be a one dimensional representation of  $k$  and extend it trivially to  $q$ . Let  $C_\lambda$  be the representation space of  $\lambda$  and define generalized Verma module  $V(\lambda)$  as follows:

$$V(\lambda) = U(g) \otimes_{U(q)} C_\lambda. \quad (4.3)$$

Jantzen [2] gave reducibility criterion for  $V(\lambda)$  using his formulas on determinants of contravariant forms. But our construction of irreducible highest weight modules  $W(\lambda)$  reproduces a part of his result.

**Corollary.** Let  $r$  be the split rank of  $G_0$  and  $t = -2i\lambda(z)/r$ . Then if  $t$  is a positive integer or satisfies  $b(t) = 0$  then  $V(\lambda)$  is reducible.

**Proof.** Here we consider only the first case but the same argument is true for any other case. As in the previous section we regard a complex number  $\lambda$  as a 1-dimensional character of  $k$ . If  $t$  is a positive integer then highest weight vector  $v^\lambda = f^\lambda$  of  $W(\lambda)$  is a polynomial function on  $p^-$ . Hence  $W(\lambda)$  becomes finite dimensional  $\mathfrak{g}$ -module and  $V(\lambda)$  is reducible.

Now we consider when  $b(\lambda) = 0$ . In this case the  $b$ -function and its defining equation (4.2) is given by the following Capelli's identity (Weyl [7]):

$$\det\left(\frac{\partial}{\partial x_{jk}}\right) \cdot \det(x_{jk})^s = s(s+1) \cdots (s+n-1) \det(x_{jk})^{s-1} \\ = b(s) \det(x_{jk})^{s-1}. \quad (4.4)$$

Suppose  $V(\lambda)$  is irreducible then  $W(\lambda)$  and  $V(\lambda)$  are isomorphic. But then by the Poincare-Birkhoff-Witt theorem  $W(\lambda)$  is isomorphic to  $U(p^-)$  as a vector space. Then

$$(-1)^n F_{1\sigma(1)} F_{2\sigma(2)} \cdots F_{n\sigma(n)} v^\lambda = \frac{\partial^n}{\partial x_{1\sigma(1)} \partial x_{2\sigma(2)} \cdots \partial x_{n\sigma(n)}} \det(x_{jk})^\lambda \quad (4.5)$$

must be linearly independent, where  $\sigma$  runs over the set of all permutations of  $\{1, 2, \dots, n\}$ . But this contradicts to the Capelli's identity. Hence  $V(\lambda)$  is also reducible in this case.

**Q.E.D.**

**Remark.** The above Corollary holds for any Hermitian symmetric pair listed in section 2. Also it is known that if  $\lambda$  is a zero of the  $b$ -function then  $W(\lambda)$  is unitarizable (Enright, Howe and Wallach [1]). Moreover these facts still remains true for the exceptional Hermitian symmetric pair

$$G_0 = \text{real form of } E_7, \quad K_0 = \text{compact form of } E_6 \times SO(2).$$

These observations are the original motivation of this research.

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